General Equations

1. Use the substitution $y = x + \frac{1}{x}$ to solve the reciprocal equation: $12x^4 - 4x^3 - 41x^2 - 4x + 12 = 0$.

2. Find two substitutions of the form $y = x - k$ which remove the third term from the equation:

$x^4 + 4x^3 - 18x^2 - 100x - 112 = 0$, and use one of them to solve the equation.

3. Solve the following equations:

(a) $\begin{cases} x^2 + y^2 + z^2 = 14 \\ xy + yz + zx = 11 \\ x + y + 2z = 9 \end{cases}$ **(b)** $\begin{cases} x + y + z = 1 \\ x^2 + y^2 + z^2 + 6xy = 0 \\ \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 0 \end{cases}$ **(c)** $\begin{cases} \frac{yz}{y+z} = 2 \\ \frac{zx}{z+x} = 3 \\ \frac{xy}{x+y} = 1 \end{cases}$

4. Solve the equations:

(a) $x^3 + y^3 = 8$; $x + y = 2$

(b) $x + y = 4$; $x^4 + y^4 = 9x^2y^2 + 1$

(c) $\frac{x}{5y+1} + \frac{y}{3x+1} = \frac{4}{15}$; $3x + 5y = 2$

(d) $x^3 = 2x + 7y$; $y^3 = 2y + 7x$

5. Prove that if $x + \frac{1}{x} = y + 1$, then $\frac{(x^2 - x + 1)^2}{x(x-1)^2} = \frac{y^2}{y-1}$.

Hence solve the equation $(x^2 - x + 1)^2 - 4x(x - 1)^2 = 0$.

6. Solve the following reciprocal equations:

(a) $14x^4 - 135x^3 + 278x^2 - 135x + 14 = 0$.

(b) $8x^4 - 42x^3 + 29x^2 + 42x + 8 = 0$.

7. Eliminate a, b, c from the equations: $x = \frac{a}{b+c},\ y = \frac{b}{c+a},\ z = \frac{c}{a+b}$.

8. Prove that the expression: $\frac{(b+c)(-x+y+z)^2 + (c+a)(x-y+z)^2 + (a+b)(x+y-z)^2}{x^2 + y^2 + z^2}$

has a constant value for all values of x, y, z which satisfy the equation: $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$.

9. If x is real, prove that:

(a) $\frac{x}{x^2 - 5x + 9}$ must lie between 1 and $-\frac{1}{11}$.

(b) $\frac{x^2 - x + 1}{x^2 + x + 1}$ must lie between 3 and $\frac{1}{3}$.

(c) $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ can have no value between 5 and 9.

10. **(a)** If the expression $3x^2 + 2Pxy + 2y^2 + 2ax - 4y + 1$ can be resolved into linear factors, prove that P must be one of the roots of the equation $P^2 + 4aP + 2a^2 + 6 = 0$.

(b) If x and y are two real quantities connected by the equation:

$$9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0,$$

show that x lies between 3 and 6, and y between 1 and 10.

11. **(a)** Find the range of values of x for which the expression:

(i) $2x^2 - 9x - 18$ is negative,

(ii) $12 - 5x - 3x^2$ is positive.

(b) Find the range of values of k for which the expression $x^2 + 6x + k$ is always positive.

12. Solve the following system of equations:

$$\begin{cases} 1 - x_1 x_2 = 0 \\ 1 - x_2 x_3 = 0 \\ \quad \ldots\ldots \\ 1 - x_{n-1} x_n = 0 \\ 1 - x_n x_1 = 0 \end{cases}$$

distinguishing between the cases n even and n odd.

13. Given that :

$$x^3 + y^3 + z^3 = a^3$$
$$x^2 + y^2 + z^2 = b^2$$
$$x + y + z = c$$

express $yz + zx + xy$ and xyz in terms of a, b, c ; hence show that x, y, z are the roots of the equation:

$$6\lambda^3 - 6c\lambda^2 - 3(b^2 - c^2)\lambda - (2a^2 - 3b^2 c + c^3) = 0$$

14. **(a)** If $x + y + z = m_1$, $xy + yz + zx = m_2$, $xyz = m_3$ and $s_i = x^i + y^i + z^i$, show that:

$s_1 = m_1$,

$s_2 = m_1^2 - 2m_2$,

$s_3 = s_2 m_1 - s_1 m_2 + 3m_3$,

$s_4 = s_3 m_1 - s_2 m_2 + s_1 m_3$,

$s_5 = s_4 m_1 - s_3 m_2 + s_2 m_3$.

(b) Solve the following system of equations:

$$\begin{cases} x + y + z = 3 \\ x^3 + y^3 + z^3 = 3 \\ x^5 + y^5 + z^5 = 3 \end{cases}$$